An Application of Universal Polynomial Chaos Expansion to Numerical Stochastic Simulations of an UWB EM Wave Propagation

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***Abstract*— In the paper a new form of universal polynomial chaos expansion, which was introduced in [1], is applied to numerical stochastic simulations of ultra-wideband electromagnetic wave propagation. It is assumed that stochastic parameters of a propagation scenario follow a Gauss distribution. The final coefficients of an expansion are analytical functions of a mean and a standard deviation of a stochastic variable (scenario parameter), which makes an expansion universal. The necessary initial coefficients have to be calculated numerically only once for a freely chosen values of polynomial basis parameters. Then these initial coefficients are used to calculate analytically the universal coefficients.**

***Index Terms*— UWB propagation, polynomial chaos expansion, stochastic simulation.**

I. Introduction

In the paper a stochastic simulation of an UWB electromagnetic (EM) wave is considered. The simulators of EM wave propagation base on different models of a propagation channel. In the paper a theoretical physical modeling of EM wave propagation in an indoor channel is taken into consideration. Among advantages of using physical modeling is a detailed insight of an influence of wave phenomena and physical parameters of a propagation channel on EM field distributions. When a parameter of a propagation channel is treated as stochastic variable we can include an imprecision of a modeling of a real channel in a simulator.

In order to make a simulation stochastic in the paper a polynomial chaos expansion is used, where coefficients of an expansion can be used to derive the first order (a mean) and the second order (a variance) stochastic moments of an EM wave distribution. In basic theory of a polynomial chaos expansion coefficients of an expansion are calculated for given stochastic distributions of propagation scenario parameters. When a mean or a standard deviation of a stochastic variable changes the coefficients have to be calculated again, which may require very much effort, especially for the case of numerical simulations where a transfer function of a channel or a ray is not given in an analytical form. In [1] it was introduced a universal polynomial chaos expansion where the expansion coefficients were derived into an analytical functions of a

mean and a standard deviation of a propagation channel parameter. The universal expansion was verified for the case of a propagation scenario whose transfer function could be found in an analytical form. The disadvantage of the presented in [1] approach is that in order to find the desired universal expansion coefficients it is often difficult to find the right realization (initial mean and standard deviation) for that approach. The reason of such situation is that when an expansion of a given transfer function in the first step of the approach in [1] was performed a Hermite polynomial basis was used, which is orthonormal in an infinite domain. However a transfer function of a propagation scenario often is defined in a limited domain, for which a transfer function has a physical meaning. In this paper this universal polynomial chaos expansion is modified in order to deal with the mentioned problem. The new results are verified in ultra-wideband frequency domain through numerical simulations of EM wave propagation on convex obstacles. A ray shooting method is used in numerical examples. The universal expansion coefficients are given also in the time domain by taking advantage of a vector fitting algorithm [3].

The paper is organized as follows. In Section II there is introduced a derivation of an ultra-wideband frequency domain and time domain universal polynomial chaos expansion coefficients using three-step algorithm by combining Jacobi and Gauss polynomials. Section III gives some numerical examples that verify the new coefficients for the case of an EM wave propagation on convex obstacles modeled by a PEC 2D elliptical cylinders. The conclusion are presented in Section IV.

II. A New Universal Polynomial Chaos Coefficients

In this section a three step algorithm for a derivation of a universal form of the new polynomial chaos coefficients is presented. It is assumed that a transfer function of a propagation scenario can be obtained during numerical

The presented work has been funded by the Polish Ministry of Science and Higher Education within the status activity task 2016 in “Development of methods for the analysis of propagation and signal processing as well as EMC issues”

simulations of an EM wave propagation. In the first step of an algorithm the transfer function is expanded using an orthonormal polynomial basis. This is the step where the only very much time consuming numerical calculations take place. During this step the initial expansion coefficients are derived which then are used to obtain the universal ones. The initial coefficients can be calculated for a freely chosen parameters of othonormal polynomial basis. In the paper a Jacobi polynomial basis is used. This approach overcomes the problem of a limited physical domain of a stochastic variable of a propagation scenario. The initial transfer function approximation can be written in the form:

$$T(\omega_n,\xi) = \sum_{k=0}^{K} a_{k,n} P_k^{\alpha 0,\beta 0}\left(f_1(\xi)\right), \tag{1}$$

where:

$$f_1(\xi) = \xi \frac{2}{b-a} - \frac{b+a}{b-a}, \tag{2}$$

while $P_k^{\alpha,\beta}(\xi)$ is a Jacobi polynomial of kth order. It should be noted that Jacobi polynomials are orthonormal in a range of their arguments from -1 to 1, while a weighting function has a Beta distribution shape. The coefficients of an expansion in (1) are calculated by formula [2]:

$$a_{k,n} = \frac{1}{\gamma_k^{\alpha,\beta}} \int_{-1}^{1} T\left(\omega_n, f_2(\xi)\right) P_k^{\alpha,\beta}(\xi) w(\alpha,\beta,\xi) d\xi \ , \tag{3}$$

where:

$$\gamma_k^{\alpha,\beta} = \int_{-1}^{1} P_k^{\alpha,\beta}(\xi) P_k^{\alpha,\beta}(\xi) w(\alpha,\beta,\xi) d\xi \ , \tag{4}$$

$$f_2(\xi) = \xi \frac{b-a}{2} + \frac{b+a}{2}, \tag{5}$$

while $w(\alpha,\beta,\xi)$ is a weighting function which describes a Beta distribution:

$$w(\alpha,\beta,\xi) = 2^{-(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} (1-\xi)^{\alpha} (1+\xi)^{\beta} \tag{6}$$

Index n in (1) and (3) corresponds to a number of a frequency sample. The arguments a and b used (1) correspond to the range of variable ξ for which the universal expansion coefficients will be derived in subsequent parts of this Section. This range must be wide enough to comprise all possible ξ values which are of our interests

As was written earlier the expansion coefficients in (1) have to be calculated only once and are tabulated. Then they are used to derive the new universal polynomial chaos expansion coefficients. It is assumed in the paper that stochastic parameter of a propagation scenario follows a Gauss distribution. Then for a given transfer function of a propagation scenario, according to polynomial chaos theory, the goal is to find an expansion [2]:

$$T(\omega_n,\xi) = \sum_{m=0}^{\infty} b_{m,n} H_m\left(\frac{\xi-\mu}{\sigma}\right), \tag{7}$$

where μ and σ are a mean and a standard deviation of a random variable ξ, respectively.
The general form of spectral coefficients for $T(\omega_n,\xi)$ can be written as follows:

$$b_{m,n} = \frac{1}{\gamma_m} \int_{-\infty}^{\infty} T(\omega_n, \sigma\xi+\mu) H_m(\xi) \frac{exp\left(\frac{-\xi^2}{2}\right)}{\sqrt{2\pi}} d\xi \ . \tag{8}$$

When the expansion in (1) and some variable transformations are performed (8) can be derived into the form:

$$b_{m,n} = \frac{1}{m!} \int_{-\infty}^{\infty} \sum_{k=0}^{K} a_{k,n} P_k^{\alpha 0,\beta 0}(g\xi+h) H_m(\xi) \frac{\exp\left(\frac{-\xi^2}{2}\right)}{\sqrt{2\pi}} d\xi \ , \tag{9}$$

where:

$$g = \frac{2\sigma}{b-a} \ , \tag{10}$$

$$h = \frac{2\mu-b-a}{b-a} \ . \tag{11}$$

In the next step a Jacobi polynomial of kth order is transformed into a sum of Hermite polynomials of maximum order k [4, 5]. As a result a Jacobi polynomial of order k and parameters $\alpha=\alpha 0$, $\beta=\beta 0$ can be written as follows:

$$P_k^{\alpha 0,\beta 0}(x) = \sum_{s=0}^{k} c_s^k H_s(x), \tag{12a}$$

$$c_s^k = \frac{1}{s!} \int_{-\infty}^{\infty} P_k^{\alpha 0,\beta 0}(x) H_s(x) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \ . \tag{12b}$$

Using expansion coefficients (12a) we can rearrange (9) and obtain:

$$b_{m,n} = \frac{1}{m!} \int_{-\infty}^{\infty} \sum_{k=0}^{K} d_{k,n} H_k(g\xi+h) H_m(\xi) \frac{\exp\left(\frac{-\xi^2}{2}\right)}{\sqrt{2\pi}} d\xi \ , \tag{13}$$

where $d_{k,n}$ is a combination of (3) and (12b) as follows:

$$d_{k,n} = \sum_{i=k}^{K} a_{i,n} c_k^{\ i} \ . \tag{14}$$

where k=0,1,2,...,K.

In the third step in order to simplify (13) the following identity is used [6]:

$$H_k(x+y)=\sum_{j=0}^{k}\binom{k}{j}\cdot y^{k-j}\cdot H_j(x)\cdot \quad (15)$$

When (15) is applied in (13) the following form of the universal expansion coefficients is obtained:

$$b_{m,n}=\frac{1}{m!}\sum_{k=0}^{K}d_{k,n}\sum_{j=0}^{k}h^{k-j}\binom{k}{j}\int_{-\infty}^{\infty}H_j(g\xi)H_m(\xi)\frac{exp\left(\frac{-\xi^2}{2}\right)}{\sqrt{2\pi}}d\xi \cdot \quad (16)$$

The infinite integral in (16) after some transformations can be partially tabulated what greatly improves its calculation efficiency:

$$b_{m,n}=\frac{1}{m!}\sum_{k=0}^{K}d_{k,n}\frac{h^k}{\left(\sqrt{1-g^{-2}}\right)^m}\sum_{j=0}^{k}\left(\sqrt{\frac{g^2-1}{h^2}}\right)^j Q(j,k,m), \quad (17)$$

where Q(j,k,m) do not depend on μ or σ therefore can be tabulated as follows. When (j - m) = 0, 2, 4, 6 ..., we have:

$$Q(j,k,m)=\frac{\frac{k!}{(k-j)!}}{\left(\frac{j-m}{2}\right)!\left(\sqrt{2}\right)^{j-m}}\cdot \quad (18)$$

while (18) is 0 for the rest values of (j - m).
When the universal expansion coefficient (17) is sampled in the ultra-wideband frequency domain, vector fitting algorithm [3] can be applied in order to transform (17) into the time domain. The approximation is performed for each d_k coefficient (14). As a result the new time domain universal polynomial expansion coefficients can be written as:

$$b(t)_m=\frac{1}{m!}\sum_{k=0}^{K}d(t)_k\frac{h^k}{\left(\sqrt{1-g^{-2}}\right)^m}\sum_{j=0}^{k}\left(\sqrt{\frac{g^2-1}{h^2}}\right)^j Q(j,k,m), \quad (19)$$

where $d(t)_k$ has the following form:

$$d(t)_k=\sum_i r_{k,i}\cdot\exp\left(p_{k,i}\cdot t\right), \quad (20)$$

where $r_{k,i}$ and $p_{k,i}$ denote residues and poles resulting from vector fitting approximation.
The new coefficients in (19) can be now easily used to calculate time domain function of a mean and a standard deviation [2].

III. SIMULATION EXAMPLES

In this section the results of universal expansion coefficients are verified by comparing them with standard Monte Carlo one in calculation of a mean and a standard deviation of exemplary $T(\omega_n,\xi)$ for frequency band 1–12 GHz with respect to a stochastic variable ξ following exemplary Gauss distributions. The transfer function is found in the process of numerical simulations of an EM wave propagation. The propagation channel consists of convex obstacles modeled by elliptical cylinders. The propagation scenario is shown in Figs. 2, 3. In numerical analysis ray shooting method is used. The transfer functions that are taken into consideration correspond to bold (brown color) rays in Figs 2, 3. The parameter of the channel that is assumed to be stochastic is shown in Fig. 3. As can be seen the axis of the ellipses can be stochastically varying in the way that horizontal axes can decrease while the vertical axes increases in the same extent. The values of horizontal and vertical axes in Figs. 2, 3 are 0,5m and 0.25m, respectively. The stochastic variable ξ in Fig. 3 is assumed to be changing within the limits 0m - 0.14m.

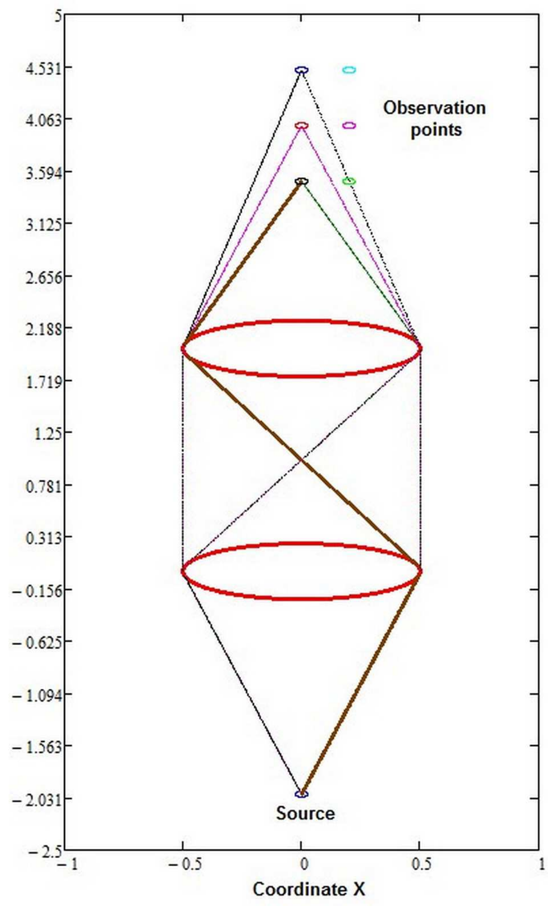

Fig. 1. Propagation scenario corresponding to the first numerical example. A bold (brown color) ray is taken into consideration.

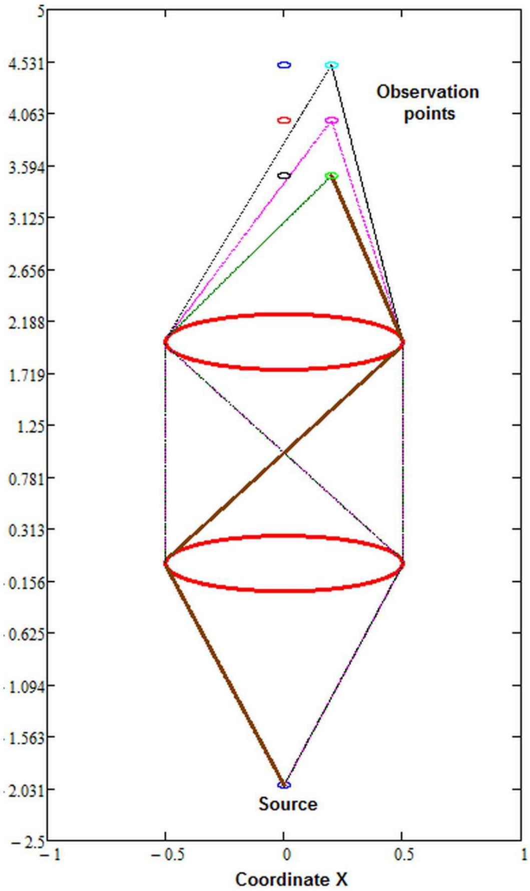

Fig. 2. Propagation scenario corresponding to the second numerical example. A Bold (brown color) ray is taken into consideration.

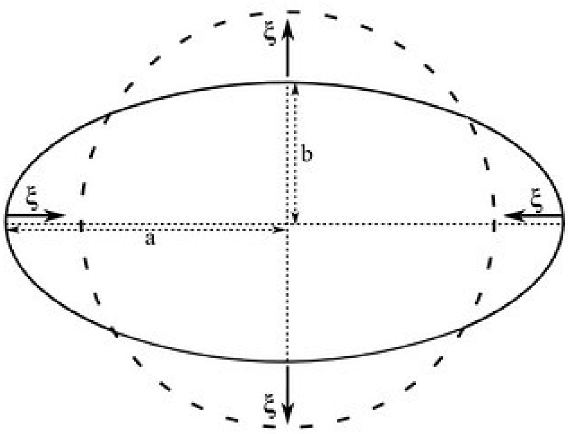

Fig. 3. An ilustration of statistical variation of axis of ellipses in Figs. 1, 2.

In the example corresponding to Fig. 1 a mean and a standard deviation of variable ξ take values 0.05m and 0.005m, respectively. The simulation results related to this example are shown in Figs. 4, 5. The second numerical example corresponds to Fig. 2. Here, a mean and a standard deviation are 0.09 and 0.09, respectively. The simulation results related to this example are shown in Figs. 6, 7.

The solid line in Figs. 4-7 corresponds to the results obtained with the new universal expansion coefficients. The dotted line and square symbol graphs correspond to Monte Carlo (MC) simulation results for different number of samples used. The graphs of mean and standard deviation characteristics are named with μ and σ, respectively. For space saving issues only real part of the functions is shown.

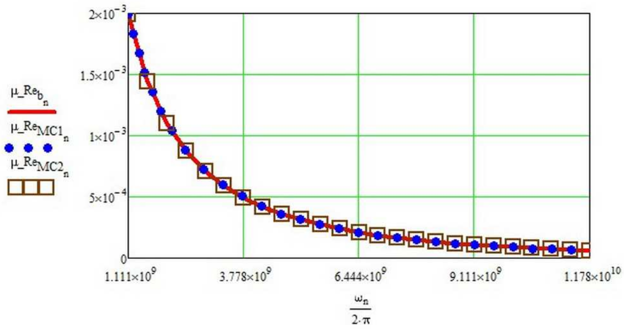

Fig. 4. Mean of a real part of a transfer function of a ray from Fig. 1 with respect to a frequency when ξ has a Gauss distribution with μ=0.05 m, σ=0.005. MC results shown with squares and circles correspond to a number of samples 100 and 1000, respectively.

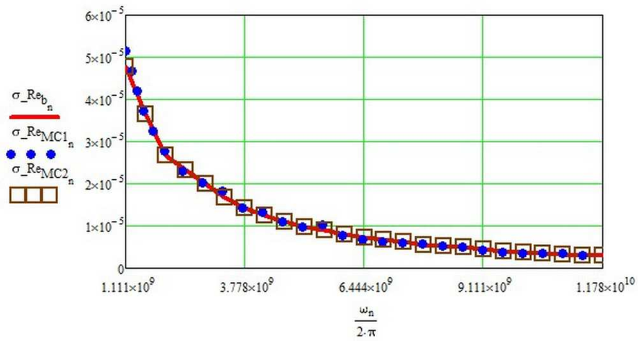

Fig. 5. Standard deviation of a real part of a transfer function of a ray from Fig. 1 with respect to a frequency when ξ has a Gauss distribution with μ=0.05 m, σ=0.005. MC results shown with squares and circles correspond to a number of samples 100 and 1000, respectively.

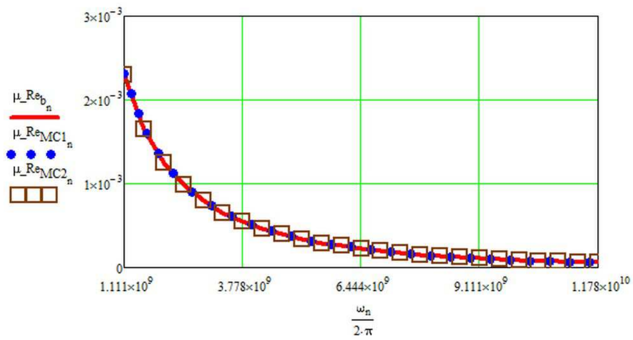

Fig. 6. Mean of a real part of a transfer function of a ray from Fig. 2 with respect to a frequency when ξ has a Gauss distribution with μ=0.09 m, σ=0.009. MC results shown with squares and circles correspond to a number of samples 100 and 1000, respectively.

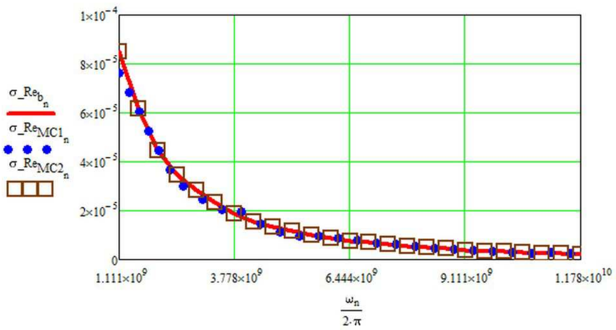

Fig. 7. Standard deviation of a real part of a transfer function of a ray from Fig. 1 with respect to a frequency when ξ has a Gauss distribution with μ=0.09 m, σ=0.009. MC results shown with squares and circles correspond to a number of samples 100 and 1000, respectively

It can be seen in Figs. 4-7 that the results obtained with the new universal expansion coefficients are very accurate and agree very well with the results of Monte Carlo method. The latter were obtained in a time that is many orders longer from that consumed by an application of the new universal expansion coefficients.

IV. CONCLUSIONS

In the paper a new universal coefficients of polynomial chaos expansion were presented. They can be used for simulation of EM wave propagation that deals with statistical analysis of EM field distribution. The coefficients were given in a frequency as well as time domain, what enables to simulate UWB pulse propagation directly in a time domain. In time domain the coefficients are given in a simple form of sums of exponential function. This feature enables low time consuming calculations of convolutions, which can be performed wholly analytically when an UWB pulse is approximated by exponential functions. The vector fitting for approximation of FFT of an UWB pulse can be used. The new universal coefficients for the case of numerical simulation of EM wave propagation were examined. The generality of the new expansion coefficients expresses in validity of the coefficients for all predefined range of possible values of a given scenario parameter.

The results presented in the paper enable to obtain stochastic distribution of a given transfer function in a very short time comparing to Monte Carlo analysis. When numerical simulations, as ray shooting or full wave analysis, are performed the time necessary for Monte Carlo method for carrying simulations with new mean an standard deviation values is many orders longer. The results obtained using the new universal expansion coefficients are very accurate what is presented for the case of exemplary stochastic variable distributions in Figs. 4-7.

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